J. Fluid Mech. (1984), vol. 148, pp. 301–318
Printed in Great Britain

Two-phase flow equations for a dilute dispersion of gas bubbles in liquid

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(Received 16 July 1983 and in revised form 13 June 1984)

Equations of motion correct to the first order of the gas concentration by volume are derived for a dispersion of gas bubbles in liquid through systematic averaging of the equations on the microlevel. First, by ensemble averaging, an expression for the average stress tensor is obtained, which is non-isotropic although the local stress tensors in the constituent phases are isotropic (viscosity is neglected). Next, by applying the same technique, the momentum-flux tensor of the entire mixture is obtained. An equation expressing the fact that the average force on a massless bubble is zero leads to a third relation. Complemented with mass-conservation equations for liquid and gas, these equations appear to constitute a completely hyperbolic system, unlike the systems with complex characteristics found previously. The characteristic speeds are calculated and shown to be related to the propagation speeds of acoustic waves and concentration waves.

1. Introduction

Mixtures of liquid and small gas bubbles occur in many industrial processes (bubble columns and centrifuges in the petrochemical industry, cloud cavitation in hydraulic systems, cooling devices of nuclear-reactor systems) and in nature (air entrained in the form of bubbles in brooks, rivers and at the surface of the ocean). A lot of attention has been paid to the problem of how to formulate equations of motion for such two-phase flows. In the case of negligible velocity differences between the phases, the equations for one-dimensional transient flow of bubbly liquids are reasonably well established (see e.g. van Wijngaarden 1968, 1972). Solutions for the propagation of acoustic disturbances, shock waves and solitons have been verified experimentally (e.g. Kuznetsov *et al.* 1976). More general equations for dispersed two-phase flows in which allowance is made for velocity differences between the phases have been formulated first in a heuristic way, but later by applying averaging techniques like time averaging (Ishii 1975), volume averaging (Nigmatulin 1979) and ensemble averaging (Buyevich & Shchelchkova 1978) to the conservation equations of the separate phases.

In §2 the equations resulting from volume averaging are briefly discussed, together with some conclusions that can be drawn from them. An important conclusion is that the equations, in their simplest form, have complex characteristics, which renders the initial-value problem ill-posed. The attempt to obtain improved equations by a more accurate description of interaction forces and other interaction effects between the phases is as yet hampered by a lack of knowledge concerning quantitative description of these effects. In §§3–6 we bypass this problem by formulating, instead of the momentum equations of the two phases separately, one averaged momentum equation for the entire mixture. An additional equation is then needed, and this is obtained by averaging the hydrodynamic force exerted on the bubbles by the surrounding fluid and using the fact that on the approximately massless bubbles the net force must vanish. Similar approaches have been applied by other authors (e.g. Voinov & Petrov 1977; Nigmatulin 1979), but, instead of averaging a relation for the force on the bubbles, they used an expression for the force on a spherical body moving in an unsteady, non-uniform flow field. It turns out, as is shown in §7, that *after averaging* a relation results that differs from those used in previous works.

The novel aspects in our work are the systematic averaging of the equations and the fact that upon inspection of the characteristics *these appear to be real.* In general, four characteristics are found (see §10). Two of these are associated with acoustic wave velocities. The remaining ones, complex in other theories, are associated with the propagation velocity of perturbations in the void fraction. These kind of waves are discussed in §9 in relation with the present theory.

2. Review of previous work

A way to formulate equations of motion for a mixture of two phases is to average the conservation equations for each of the phases over the volume occupied by the pertinent phase in a suitably chosen averaging volume. In the averaging process, fluctuations disappear and equations for average or mean quantities are obtained. This method, which has its specific difficulties and pitfalls because of the problem of dealing correctly with discontinuities at the boundaries between the two phases, has been applied, for example by Nigmatulin (1979), in the following way.

Let us denote the local density by ρ, velocity by $\boldsymbol{u}$, stress by $\boldsymbol{\sigma}$ and volume concentration of phase k by α_k. Averaging over the volume occupied by phase k is indicated with an overbar and the subscript k (e.g. $\bar{\boldsymbol{u}}_k$). The equations for conservation of mass and momentum for phase k, in the absence of mass transfer between the phases and excluding gravity and other external forces, are as follows:

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\alpha_k(\bar{\rho})_k+\boldsymbol{\nabla}\cdot\alpha_k(\overline{\rho\boldsymbol{u}})_k = 0, \\ &\frac{\partial}{\partial t}\alpha_k(\overline{\rho\boldsymbol{u}})_k+\boldsymbol{\nabla}\cdot\alpha_k(\overline{\rho\boldsymbol{u}\boldsymbol{u}})_k = \boldsymbol{\nabla}\cdot\alpha_k(\bar{\boldsymbol{\sigma}})_k+\frac{1}{V}\int_{A_\mathrm{I}}\boldsymbol{\sigma}\cdot\boldsymbol{n}_k\,\mathrm{d}A. \end{aligned}\right\} \tag{2.1}$$

The integrals over the interfaces A_I inside the averaging volume V represent the interaction forces between the phases. The normals $\boldsymbol{n}_k$ are directed outward with respect to phase k.

It is common practice to write the momentum-flux tensors as

$$\overline{((\rho\boldsymbol{u})\,\boldsymbol{u})}_k = (\overline{\rho\boldsymbol{u}})_k\,\bar{\boldsymbol{u}}_k+(\overline{(\rho\boldsymbol{u})'\,\boldsymbol{u}'})_k, \tag{2.2}$$

where the primes denote fluctuations about the averages. The fluctuation terms can, as in the theory of turbulence (Reynolds stresses), conveniently be combined with the average actual stress tensors. When density variations within a single phase are discarded, the conservation equations become

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\alpha_k\,\rho_k+\boldsymbol{\nabla}\cdot\alpha_k\,\rho_k\,\bar{\boldsymbol{u}}_k = 0, \\ &\alpha_k\,\rho_k\left[\frac{\partial}{\partial t}\bar{\boldsymbol{u}}_k+\bar{\boldsymbol{u}}_k\cdot\nabla\bar{\boldsymbol{u}}_k\right] = \boldsymbol{\nabla}\cdot\alpha_k[\bar{\boldsymbol{\sigma}}_k-(\overline{(\rho\boldsymbol{u})'\,\boldsymbol{u}'})_k]+\frac{1}{V}\int_{A_\mathrm{I}}\boldsymbol{\sigma}\cdot\boldsymbol{n}_k\,\mathrm{d}A. \end{aligned}\right\} \tag{2.3}$$

For further progress, specification of the stresses and the interaction forces is required. The interaction forces may include viscous forces, inertia or virtual mass forces, and surface tension. Usually one employs general forms of constitutive equations together with suitably chosen coefficients based on experimental results (see e.g. Drew & Lahey 1979); however the correct formulation of these terms is still a subject of discussion in the two-phase-flow literature.

The simplest set of equations is obtained when the Reynolds stresses, viscous and other interaction terms, except for an isotropic pressure term, are omitted, and the averaged pressures of the phases are assumed to be equal. For one-dimensional flow of gas (subscript g) and liquid (subscript ℓ), the equations are in that case

$$\frac{\partial}{\partial t}\alpha\rho_g+\frac{\partial}{\partial x}\alpha\rho_g U_g = 0, \tag{2.4}$$

$$\frac{\partial}{\partial t}(1-\alpha)\rho_\ell+\frac{\partial}{\partial x}(1-\alpha)\rho_\ell U_\ell = 0, \tag{2.5}$$

$$\alpha\rho_g\left(\frac{\partial U_g}{\partial t}+U_g\frac{\partial U_g}{\partial x}\right)+\alpha\frac{\partial p}{\partial x} = 0, \tag{2.6}$$

$$(1-\alpha)\rho_\ell\left(\frac{\partial U_\ell}{\partial t}+U_\ell\frac{\partial U_\ell}{\partial x}\right)+(1-\alpha)\frac{\partial p}{\partial x} = 0. \tag{2.7}$$

Here U_g and U_ℓ are the averaged gas and liquid velocities, p is the averaged pressure and α is the volumetric fraction of the gas phase, or void fraction.

This set of equations has been discussed by many authors (van Wijngaarden 1976*a*; Stuhmiller 1977). A disturbing feature is that the set has two real and two complex characteristics, and is therefore not completely hyperbolic, which means that the initial-value problem is ill-posed. Any correct numerical scheme for solving such a set of equations will therefore develop instabilities. On the other hand, there is experimental evidence (see §9) that in a dilute mixture of liquid and gas bubbles stable wavelike disturbances are possible. There have been numerous attempts to overcome this difficulty by introducing additional terms. Although much work has been done, it has as yet not been possible to derive a set of equations with real characteristics from first principles.

As mentioned earlier in this paper, the formulation of the phase-interaction forces can be bypassed if we do not try to find momentum equations for the separate phases but average the local momentum equations over the entire mixture. Such an approach is also used in the theory of suspensions of particles in viscous liquids by Batchelor and associates (Batchelor 1970, 1974; Batchelor & Green 1972), and in the next sections we will apply a technique similar to that used by Batchelor (1970) to a mixture of liquid and small spherical gas bubbles.

3. The bulk stress tensor

We consider unsteady and spatially inhomogeneous flow of gas bubbles in a slightly viscous incompressible liquid. As a result of external forces, e.g. gravity, the bubbles move with respect to the liquid. We assume that the Reynolds number for the relative translational motion of a single bubble with respect to the bulk motion is large. At the same time, surface tension is supposed to be large enough to keep the bubbles approximately spherical.

We want to formulate for this dispersion equations of motion on a continuum basis, and with a view to that we make a distinction between three lengthscales.

(i) The *microscale*. For this the mean particle distance, proportional to $n^{-\frac{1}{3}}$, where n is the number density, is a useful quantity.

(ii) The *macroscale*, L say. This is the distance over which mean quantities like $\boldsymbol{U}_g$ or $\boldsymbol{U}_\ell$ vary significantly. The precise definition of 'mean' is given later. For example L can be the length of a wave progressing through the mixture. In the following we want to derive equations describing relations between $\boldsymbol{U}_g$, $\boldsymbol{U}_\ell$ etc. and their spatial derivatives. Fluctuations on the microscale are taken into account only insofar as they produce effects on the macroscale. In order to accomplish this we introduce

(iii) the *mesoscale*, l say, which is small with respect to L but large with respect to $n^{-\frac{1}{3}}$. The scales are schematically indicated in figure 1.

If we consider an element of the mixture on the mesoscale, and the motion of the bubbles in it, the simplest description of the motion of an individual bubble is that of a bubble in translational motion relative to the surrounding liquid. Therefore, before embarking on the averaging procedure, we pause to discuss uniform flow around a gas bubble.

With a relative velocity in water of 20 cm/s the Reynolds number for such a flow is 400 for a bubble with a radius of 1 mm. At a Reynolds number of this magnitude, inertia forces dominate over viscous forces, and when the liquid is free of surface-active agents the flow is amenable to calculation. The results of Moore (1963) and, more recently, of Pham (1982) give the following picture of the flow around a spherical gas bubble at moderate Reynolds numbers. The main part of the velocity and pressure distribution is that associated with the potential flow around the bubble. This satisfies the condition of no relative motion in the direction of the normal to the surface, but not the condition of vanishing tangential stress.† This is reduced from its value in the potential flow to zero in a boundary layer with a thickness of order $Re^{-\frac{1}{2}}$, where Re is the Reynolds number based on the free-stream velocity. This Reynolds number is supposed here to be large enough to make the associated velocity perturbation, which is of order $Re^{-\frac{1}{2}}$, negligibly small. The concept of a boundary layer cannot be continued till the rear stagnation point. It can be shown, however, that also at the rear the deviation of velocities and pressures from those of the primary potential flow vanish at high Reynolds numbers. Accordingly they will be disregarded in the calculation of the bulk stress tensor.

Regarding the role of viscosity, two more remarks should be made. The first concerns the wake behind the bubble. Its width is of order $Re^{-\frac{1}{4}}$ (Moore 1963), and the associated velocity perturbation is of order $Re^{-\frac{1}{2}}$. Since we suppose Re to be large, we will neglect the velocity perturbation due to the wake just as we neglect that of the boundary layer. There is an important difference here with the flow around a gas bubble in an impure liquid, where a no-slip condition applies at the interface. In that case there is at high Reynolds numbers a wake of finite width and a velocity perturbation of order one at the rear of the bubble.‡ At the present state of knowledge about the flow around rigid bodies at high Reynolds numbers it is not possible to predict this velocity perturbation. We will therefore restrict ourselves to the case of pure liquids, admitting that a similar theory for contaminated liquids would be highly desirable since these occur frequently in practice.

A second remark concerns the frictional force on the bubble. Although the pressure correction with respect to the potential flow cannot be calculated near the rear

† This would be the exact boundary condition for an empty bubble, but may also serve as boundary condition for a gas bubble in view of the small viscosity with respect to that of water.

‡ In a mixture the significance of wakes is probably less than in a free stream because the presence of neighbouring bodies suppresses the wake considerably.

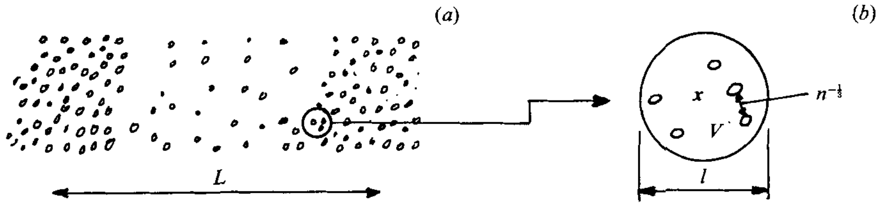

FIGURE 1. Lengthscales in bubbly liquid: (*a*) macroscale L; (*b*) microscale $n^{-\frac{1}{3}}$ and mesoscale ℓ.

stagnation point, an expression for the viscous resistance force can be obtained from a calculation of the viscous dissipation in the fluid outside the boundary layer as first shown by Levich (see Batchelor 1967, p. 367), or from momentum considerations (Moore 1963). In §9 we will use the result for the viscous drag $\boldsymbol{D} = 12\pi\mu R\,\Delta\boldsymbol{u}$, where μ is the viscosity of the liquid and $\Delta\boldsymbol{u}$ the velocity difference between the free stream and the bubble.

After this excursion to the flow around a single bubble, we return to the bubbly flow with the various scales as depicted in figure 1. We consider a volume V with a linear dimension of the order of the mesoscale l. This volume is centred about the macroscopic vector position $\boldsymbol{x}$ and contains a large number N of gas bubbles, with equal radius $R(\boldsymbol{x}, t)$. The relation between n, V and N is therefore $n = N/V$. Since the same macroscopic conditions can correspond to many different realizations of the positions of N bubbles inside V, and since a large number of realizations of sphere positions constitute an ensemble, it is appropriate to define the average value of a quantity as the integral of the local value of that quantity over the ensemble of all possible configurations of the N bubbles in V. Let the probability of finding a certain configuration C_N of N bubbles in V be $P(C_N)\,\mathrm{d}C_N$. We now define the ensemble-averaged stress tensor $\langle\boldsymbol{\sigma}\rangle$ by

$$\langle\boldsymbol{\sigma}\rangle = \frac{1}{N!}\int \boldsymbol{\sigma}(\boldsymbol{x}, C_N)\,P(C_N)\,\mathrm{d}C_N, \tag{3.1}$$

where $\boldsymbol{x}$ may be in one realization in liquid and in another realization in gas.

In our continuum approach mean quantities defined in this way, and depending on $\boldsymbol{x}$ and t, are constant on the mesoscale, which, as we have assumed, is small compared with the macroscale. On the other hand there are a large number of particles, in our case bubbles, in V, because the mesoscale l is large compared with the interparticle distance. There is already a rich literature on the calculation of average quantities, defined as above, for the case in which the flow around individual particles is dominated by viscous forces, as for example in the sedimentation of small particles in liquid under gravity. The theory of viscous suspensions has been reviewed, for example, by Batchelor (1974). In order to make an optimal use of the results obtained by Batchelor in a series of papers beginning with Batchelor (1970), we will follow his approach, which employs ensemble averaging but uses at certain stages the equivalence with volume averaging which holds for a statistically homogeneous medium. However, a formulation either completely in terms of volume averaging or of ensemble averaging is possible, in principle. We will henceforth denote both types of averaging, when used, by $\langle\;\rangle$.

Making use now of the equivalence with volume averaging, we write (3.1) as a volume average, following Batchelor (1970). We divide the volume V into three parts: (i) the volume V_g occupied by gas; (ii) a vanishingly small volume ΣV_ϵ containing the

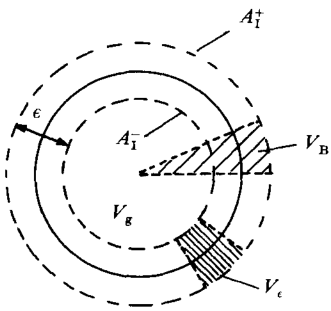

FIGURE 2. Bubble with volume V_B, which includes an interfacial layer with thickness ϵ and volume V_ϵ, bounded by the surfaces A_I^+ and A_I^-.

interfacial layers between gas and liquid; and (iii) the remainder of V, wholly occupied by liquid. If we indicate the sum $V_g+\Sigma V_\epsilon$ by V_B, the volume of (iii) is $V-\Sigma V_B$. This partition is shown schematically in figure 2.

With this specification of the volumes involved,

$$\langle\sigma_{ij}\rangle = \frac{1}{V}\int_{V-\Sigma V_B}\sigma_{ij}\,\mathrm{d}V+\frac{1}{V}\Sigma\int_{V_B}\sigma_{ij}\,\mathrm{d}V. \tag{3.2}$$

Recalling that we have denoted the mean number density by

$$n=\frac{N}{V}=\frac{3\alpha}{4\pi R^3},$$

we write

$$\frac{1}{V}\Sigma\int_{V_B}\sigma_{ij}\,\mathrm{d}V = n\left\langle\int_{V_B}\sigma_{ij}\,\mathrm{d}V\right\rangle, \tag{3.3}$$

where the integration is over the volume of a test sphere or reference sphere. The value of this integral will in general depend on the position of other spheres, and the average in (3.3) is an ensemble average over the possible realizations of sphere centres in V. This interpretation is particularly helpful in examining the effects of hydrodynamic interaction, as is excellently demonstrated for the case of a viscous suspension by Batchelor & Green (1972).

In order to evaluate the integral in (3.3), we introduce the position vector $\boldsymbol{r}$ of a point with respect to the centre $\boldsymbol{x}$ of the test sphere. Transformation of the volume integral on the right-hand side of (3.3) into a surface integral gives

$$\int_{V_B}\sigma_{ij}\,\mathrm{d}V = \int_{A_B}\sigma_{ik}\,r_j\,n_k\,\mathrm{d}A-\int_{V_B}\frac{\partial\sigma_{ik}}{\partial r_k}r_j\,\mathrm{d}V. \tag{3.4}$$

The surface A_B lies completely in liquid (i.e. outside the interfacial layer), and, with pressure p in the liquid,

$$\int_{A_B}\sigma_{ik}\,r_j\,n_k\,\mathrm{d}A = -\int_{A_B}p\delta_{ik}\,r_j\,n_k\,\mathrm{d}A = -\int_{A_B}pr_j\,n_i\,\mathrm{d}A, \tag{3.5}$$

where δ_{ik} is the Kronecker delta. A hydrostatic head is not included in p. Gravity will be introduced in §6, where the momentum equation is formulated.

With regard to the second integral on the right-hand side of (3.4), the divergence of the stress, as shown by Batchelor (1970), is regular in V_B (although the stress itself may be large in the interfacial layer), and when we neglect the inertia of the gas or vapour within the bubbles this integral vanishes. Hence, from (3.3)–(3.5),

$$\frac{1}{V}\Sigma\int_{V_{\mathrm{B}}}\sigma_{ij}\,\mathrm{d}V = -n\left\langle\int_{A_{\mathrm{B}}} pr_j n_i\,\mathrm{d}A\right\rangle. \tag{3.6}$$

The volume $V-\Sigma V_{\mathrm{B}}$ is occupied by liquid, and the first integral on the right-hand side of (3.2) may be written as

$$\frac{1}{V}\int_{V-\Sigma V_{\mathrm{B}}}\sigma_{ij}\,\mathrm{d}V = (1-\alpha)\langle\sigma_{ij}\rangle_\ell = -(1-\alpha)\langle p\rangle_\ell\,\delta_{ij}. \tag{3.7}$$

Thus the bulk stress (3.1) is the sum of the right-hand sides of (3.6) and (3.7), i.e.

$$\langle\sigma_{ij}\rangle = -(1-\alpha)\langle p\rangle_\ell\,\delta_{ij} - n\left\langle\int_{A_{\mathrm{B}}} pr_j n_i\,\mathrm{d}A\right\rangle. \tag{3.8}$$

The pressure p in the integral in (3.8) results from the motion of the test sphere itself and from the motion of the other spheres in the volume V. Its determination requires solution of Laplace's equation with appropriate boundary conditions on all spheres. This is impossible to achieve, but for dilute mixtures one can in a first approximation consider the test bubble to be isolated in an infinite liquid. The probability that if there is a bubble at $\boldsymbol{x}+\boldsymbol{r}$ there is another one located at a distance of the order of the radius R, is proportional to α, so that interactions with one other bubble are involved in the calculation of $\langle\sigma_{ij}\rangle$ in the next approximation. In the first approximation to which we restrict ourselves here, the integral in (3.8) has the same value for each realization, and we have only to calculate its value in the case of a uniform flow along the test sphere. Far from the test sphere, moving at velocity $\boldsymbol{U}_{\mathrm{g}}$, the liquid velocity and pressure are uniform, with values

$$\boldsymbol{u} = \boldsymbol{U}_0(\boldsymbol{x},t), \quad p = \langle p(\boldsymbol{x},t)\rangle \tag{3.9}$$

respectively.

The test bubble is always spherical. However, its radius may change, and the rate of change of the radius is denoted by $\mathrm{d}R/\mathrm{d}t$. Mass conservation for any bubble, together with (2.4) and (3.9), gives

$$\nabla\cdot\boldsymbol{U}_0 = n\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{4}{3}\pi R^3).$$

The hydrodynamic potential ϕ is, with $\boldsymbol{x}$ and $\boldsymbol{r}$ as shown in figure 3,

$$\phi(\boldsymbol{x}+\boldsymbol{r},t) = \boldsymbol{U}_0(\boldsymbol{x},t)\cdot\boldsymbol{r} - \frac{R^2\dfrac{\mathrm{d}R}{\mathrm{d}t}}{r} - \tfrac{1}{2}R^3\{\boldsymbol{U}_0(\boldsymbol{x},t)-\boldsymbol{U}_{\mathrm{g}}(\boldsymbol{x},t)\}\cdot\nabla_r\frac{1}{r}. \tag{3.10}$$

The pressure $p(\boldsymbol{x}+\boldsymbol{r},t)$ follows from Bernoulli's theorem, and is, with ρ_ℓ denoting the density of the liquid,

$$\frac{p(\boldsymbol{x}+\boldsymbol{r},t)}{\rho_\ell} = -\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\{|\nabla_r\phi|\}^2 + \boldsymbol{U}_{\mathrm{g}}\cdot\nabla_r\phi + \frac{\langle p\rangle(\boldsymbol{x},t)}{\rho_\ell}, \tag{3.11}$$

where the time derivative $\partial\phi/\partial t$ is at constant $\boldsymbol{r}$. From (3.10) and (3.11) we find

$$\begin{aligned}-\int_{r=R} pr_j n_i\,\mathrm{d}A = {}& -\tfrac{4}{3}\pi R^3\left[\langle p\rangle + \tfrac{3}{2}\rho_\ell\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + \rho_\ell R\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - \tfrac{1}{4}\rho_\ell(U_{\mathrm{g}}-U_0)_m(U_{\mathrm{g}}-U_0)_m\right]\delta_{ij}\\ &+\tfrac{4}{3}\pi R^3\rho_\ell[\tfrac{3}{20}(U_{\mathrm{g}}-U_0)_m(U_{\mathrm{g}}-U_0)_m\delta_{ij} - \tfrac{9}{20}(U_{\mathrm{g}}-U_0)_i(U_{\mathrm{g}}-U_0)_j].\end{aligned} \tag{3.12}$$

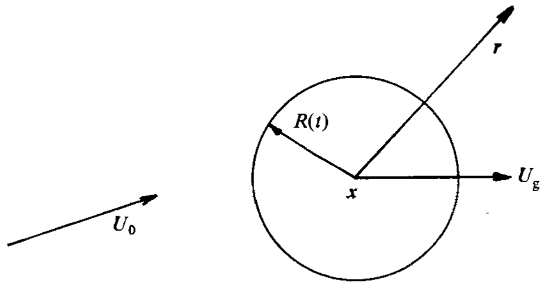

FIGURE 3. Reference sphere at $\boldsymbol{x}$ with radius $R(t)$, moving with velocity $\boldsymbol{U}_g$ in an unbounded fluid with velocity $\boldsymbol{U}_0$ at infinity.

Combining (3.8) and (3.12) gives for the bulk stress

$$\langle \boldsymbol{\sigma} \rangle = -\left[(1-\alpha)\langle p \rangle_\ell + \alpha \langle p \rangle + \tfrac{3}{2}\rho_\ell \alpha \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + \rho_\ell \alpha R \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} - \tfrac{1}{4}\rho_\ell \alpha \{|\boldsymbol{U}_g - \boldsymbol{U}_0|\}^2 \right] \boldsymbol{I}$$
$$+\rho_\ell \alpha [\tfrac{3}{20}\{|\boldsymbol{U}_g - \boldsymbol{U}_0|\}^2 \boldsymbol{I} - \tfrac{9}{20}(\boldsymbol{U}_g - \boldsymbol{U}_0)(\boldsymbol{U}_g - \boldsymbol{U}_0)], \quad (3.13)$$

where $\boldsymbol{I}$ denotes the second-order unit tensor.

By definition, the bulk pressure $\langle p \rangle$ is related to $\langle \sigma_{ij} \rangle$ by

$$\langle p \rangle = -\tfrac{1}{3}\langle \sigma_{ii} \rangle,$$

which allows us to write (3.13), accurate to $O(\alpha)$, as

$$\langle \boldsymbol{\sigma} \rangle = -\langle p \rangle \boldsymbol{I} + \rho_\ell \alpha [\tfrac{3}{20}\{|\boldsymbol{U}_g - \boldsymbol{U}_0|\}^2 \boldsymbol{I} - \tfrac{9}{20}(\boldsymbol{U}_g - \boldsymbol{U}_0)(\boldsymbol{U}_g - \boldsymbol{U}_0)]. \quad (3.14)$$

Furthermore, from (3.13) and (3.14), it follows that the pressure averaged over the liquid is related to the bulk pressure by

$$\langle p \rangle_\ell = \langle p \rangle - \alpha \rho_\ell \left[\frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + R\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} - \tfrac{1}{4}\{|\boldsymbol{U}_g - \boldsymbol{U}_0|\}^2 \right]. \quad (3.15)$$

The result (3.14) is particularly interesting. It appears that, although the local stress tensor in each of the phases is isotropic, the average stress tensor contains, owing to the relative velocity of the phases, a non-isotropic part. From (3.15) we see that the contribution $\langle p \rangle - \langle p \rangle_\ell$ of the bubbles to the bulk pressure consists of a part that stems from relative translational motion and a part that is due to the volume changes of the bubbles.

4. The phase-averaged pressures

In §3 the bulk stress has been obtained by averaging over the entire dispersion. Sometimes it is convenient to take averages over the separate phases. The relation between the bulk pressure $\langle p \rangle$ and the average pressure $\langle p \rangle_\ell$ in the liquid is given by (3.15). To obtain the relation between the average gas pressure $\langle p \rangle_g$ and $\langle p \rangle$ we have to make a partition different from (3.1). V_B consists of a volume V_ϵ in which the interface is included and a volume $V_B - V_\epsilon$ completely occupied by gas. At the fluid side V_ϵ is bounded by a closed surface A_I^+, and at the gas side by a surface A_I^-. As in (3.4) we write

$$\int_{V_\epsilon} \sigma_{ij}\,\mathrm{d}V = \left(\int_{A_I^+} - \int_{A_I^-}\right) \sigma_{ik} r_j n_k \,\mathrm{d}A - \int_{V_\epsilon} \frac{\partial \sigma_{ik}}{\partial r_k} r_j \,\mathrm{d}V. \quad (4.1)$$

We take $\epsilon \to 0$ and, again, since the divergence of the stress is regular inside V_ϵ, even if in the interface a surface stress is present, we obtain (for a detailed account see the appendix in Batchelor 1970)

$$\lim_{\epsilon\to 0}\int_{V_\epsilon} \sigma_{ij}\,\mathrm{d}V = \int_{A_{\mathrm{I}}} \gamma r_j n_i \nabla\cdot\boldsymbol{n}_{\mathrm{g}}\,\mathrm{d}A, \tag{4.2}$$

where γ is the coefficient of surface tension and $\gamma n_i \nabla\cdot\boldsymbol{n}_{\mathrm{g}}$ is the jump in normal stress across the interface due to surface tension. The latter consists of an average, $2\gamma/R$, and a varying part $[p]_{\mathrm{I}}$ due to (slight) departures of the spherical form as a consequence of the motion of the bubbles:

$$\int_{A_{\mathrm{I}}} \gamma r_j n_i \nabla\cdot\boldsymbol{n}_{\mathrm{g}}\,\mathrm{d}A = \tfrac{8}{3}\pi\gamma R^2\delta_{ij} - \int_{A_{\mathrm{I}}} [p]'_{\mathrm{I}}\, r_j n_j\,\mathrm{d}A. \tag{4.3}$$

Inside the bubbles the pressure is, owing to the small gas density, uniform and equal to $p_{\mathrm{g}} = \langle p\rangle_{\mathrm{g}}$. At the outside of the interface the pressure is equal to the liquid pressure. The integral in (4.3) is therefore due to deviations of the local liquid pressure from the average pressure over this surface:

$$\int_{A_{\mathrm{I}}} [p]'_{\mathrm{I}}\, r_j n_i\,\mathrm{d}A = \int_{A_{\mathrm{I}}} (p - \langle p\rangle_{A_{\mathrm{I}}})\, r_j n_i\,\mathrm{d}A. \tag{4.4}$$

The bulk stress tensor is now written as

$$\langle\sigma_{ij}\rangle = (1-\alpha)\langle\sigma_{ij}\rangle_\ell + \alpha\langle\sigma_{ij}\rangle_{\mathrm{g}} + \lim_{\epsilon\to 0} n\left\langle \int_{V_\epsilon} \sigma_{ij}\,\mathrm{d}V\right\rangle.$$

Using (3.11) and (4.2)–(4.4), we finally obtain

$$\langle\boldsymbol{\sigma}\rangle = -\left[(1-\alpha)\langle p\rangle_\ell + \alpha\langle p\rangle_{\mathrm{g}} - \frac{2\gamma\alpha}{R}\right]\boldsymbol{I}$$
$$+\alpha\rho_\ell[\tfrac{3}{20}\{|\boldsymbol{U}_{\mathrm{g}} - \boldsymbol{U}_0|\}^2\boldsymbol{I} - \tfrac{9}{20}(\boldsymbol{U}_{\mathrm{g}} - \boldsymbol{U}_0)(\boldsymbol{U}_{\mathrm{g}} - \boldsymbol{U}_0)]. \tag{4.5}$$

Comparison with (3.14) shows that

$$\langle p\rangle = (1-\alpha)\langle p\rangle_\ell + \alpha\langle p\rangle_{\mathrm{g}} - \frac{2\gamma}{R}\alpha, \tag{4.6}$$

or, with the help of (3.15),

$$\langle p\rangle_{\mathrm{g}} = \langle p\rangle + \frac{2\gamma}{R} + \rho_\ell\left[\frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + R\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - \tfrac{1}{4}\{|\boldsymbol{U}_{\mathrm{g}} - \boldsymbol{U}_0|\}^2\right]. \tag{4.7}$$

5. The momentum-flux tensor, including Reynolds stresses

The momentum of the mixture is entirely due to the liquid, because the contribution of the gas may reasonably be neglected. We therefore write the average momentum flux as

$$\boldsymbol{M} = \frac{\rho_\ell}{V}\int_{V-\Sigma V_{\mathrm{B}}} \boldsymbol{u}\boldsymbol{u}\,\mathrm{d}V,$$

where V_{B} has the same meaning as in the previous sections. We divide the local velocity into the average velocity $\boldsymbol{U}_0$ and a fluctuating part $\boldsymbol{u}'$. Inserting this in the above expression it follows that

$$\boldsymbol{M} = \rho_l(1-\alpha)\boldsymbol{U}_0\boldsymbol{U}_0 + \rho_\ell(1-\alpha)[\boldsymbol{U}_0(\boldsymbol{U}_{\mathrm{g}} - \boldsymbol{U}_0) + (\boldsymbol{U}_{\mathrm{g}} - \boldsymbol{U}_0)\boldsymbol{U}_0] + \frac{\rho_\ell}{V}\int_{V-\Sigma V_{\mathrm{B}}} \boldsymbol{u}'\boldsymbol{u}'\,\mathrm{d}V. \tag{5.1}$$

This expression has been considered in van Wijngaarden (1976*b*) for the case of rigid spheres. A calculation allowing for volume changes of the bubbles starts with the calculation of $\boldsymbol{u}'$ from the potential (3.10):

$$\boldsymbol{u}'(\boldsymbol{x}+\boldsymbol{r},t) = \nabla_r\phi-\boldsymbol{U}_0 = \frac{R^2\dfrac{\mathrm{d}R}{\mathrm{d}t}}{r^3}\boldsymbol{r}-\frac{(\boldsymbol{U}_g-\boldsymbol{U}_0)\,R^3}{2r^3}+\frac{3}{2}\frac{(\boldsymbol{U}_g-\boldsymbol{U}_0)\,R^3\cdot\boldsymbol{r}}{r^5}\boldsymbol{r}. \quad (5.2)$$

Again, as in the previous sections, we assume that the suspension is so dilute that interactions are negligible and within that approximation (which is accurate in the first order of the void fraction α) it is legitimate to insert (5.2) in (5.1) and to integrate over all space outside the bubble:

$$\frac{1}{V}\int_{V-\Sigma V_{\mathrm{B}}}\boldsymbol{u}'\boldsymbol{u}'\,\mathrm{d}V = n\int_{r\geqslant R(t)}\boldsymbol{u}'\boldsymbol{u}'\,\mathrm{d}\boldsymbol{r}+O(\alpha^2).$$

Making use of (5.2) and carrying out the integration gives

$$\int_{r\geqslant R(t)}\boldsymbol{u}'\boldsymbol{u}'\,\mathrm{d}\boldsymbol{r} = \tfrac{4}{3}\pi R^3\left[\left\{\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2+\tfrac{3}{20}\{|\boldsymbol{U}_g-\boldsymbol{U}_0|\}^2\right\}\boldsymbol{I}+\tfrac{1}{20}(\boldsymbol{U}_g-\boldsymbol{U}_0)(\boldsymbol{U}_g-\boldsymbol{U}_0)\right]. \quad (5.3)$$

Compared with van Wijngaarden (1976*b*) there is an additional term here, $(\mathrm{d}R/\mathrm{d}t)^2\boldsymbol{I}$, due to the rate of change of bubble volume.

Since we are dealing with the liquid, the gas carrying negligible momentum, it is useful to express (5.1) in terms of $\boldsymbol{U}_\ell$ and $\boldsymbol{U}_g$, rather than $\boldsymbol{U}_0$ and $\boldsymbol{U}_g$. The relation between them is

$$\boldsymbol{U}_0 = (1-\alpha)\,\boldsymbol{U}_\ell+\alpha\boldsymbol{U}_g. \quad (5.4)$$

Inserting (5.3) into (5.1) and making use of (5.4), we obtain

$$\frac{\boldsymbol{M}}{\rho_\ell} = (1-\alpha)\,\boldsymbol{U}_\ell\,\boldsymbol{U}_\ell+\alpha\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2+\tfrac{3}{20}\{|\boldsymbol{U}_g-U_0|\}^2\right]\boldsymbol{I}+\tfrac{1}{20}\alpha(\boldsymbol{U}_g-\boldsymbol{U}_0)(\boldsymbol{U}_g-\boldsymbol{U}_0).$$

From (2.3) and (5.5) the Reynolds stresses in the liquid can be obtained. The result is

$$\langle(\rho\boldsymbol{u})'\boldsymbol{u}'\rangle = \alpha\rho_\ell\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2+\tfrac{3}{20}\{|\boldsymbol{U}_g-\boldsymbol{U}_0|\}^2\right]\boldsymbol{I}+\tfrac{1}{20}\alpha\rho_\ell(\boldsymbol{U}_g-\boldsymbol{U}_0)(\boldsymbol{U}_g-\boldsymbol{U}_0).$$

6. The momentum equation for the dispersion

In the foregoing sections we have obtained an expression, (3.14), for the average stress tensor in the dispersion as well as an expression, (5.5), for the momentum flux. The equation of motion is obtained by equating the rate of change of momentum to the sum of the divergence of the stress tensor and external forces. Such external forces are permitted that are constant on the mesoscale. Gravity is such a force, and, by using (3.14) and (5.5), we obtain

$$\frac{\partial}{\partial t}\rho_\ell(1-\alpha)\,\boldsymbol{U}_\ell+\nabla\cdot\rho_\ell(1-\alpha)\,\boldsymbol{U}_\ell\,\boldsymbol{U}_\ell$$
$$= -\nabla\langle p\rangle+\rho_\ell(1-\alpha)\,\boldsymbol{g}-\nabla\cdot\rho_\ell\,\alpha\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\boldsymbol{I}+\tfrac{1}{2}(\boldsymbol{U}_g-\boldsymbol{U}_0)(\boldsymbol{U}_g-\boldsymbol{U}_0)\right]+O(\alpha^2), \quad (6.1)$$

where $\boldsymbol{g}$ denotes the acceleration due to gravity. The terms on the left-hand side are the well-known local and convective acceleration terms pertaining to the average liquid velocity. On the right-hand side we have first the gradient of the bulk pressure, then gravity, and finally an expression which is completely due to the relative motion.

It is appropriate to compare (6.1) with results obtained by others. A momentum equation for a mixture without relative translational motion was given in van Wijngaarden (1968). No specific averaging technique was used. The purpose was to bring out the dispersion of acoustic waves. In that connection (4.7) plays an important part, in which for small $\mathrm{d}R/\mathrm{d}t$ and in the absence of relative translational motion and surface tension only the term $\rho_\ell R\,\mathrm{d}^2R/\mathrm{d}t^2$ survives. If we take $\boldsymbol{U}_\mathrm{g}-\boldsymbol{U}_0=0$ then there is still with respect to usual presentations the term $\nabla\cdot\rho_\ell\,\alpha(\mathrm{d}R/\mathrm{d}t)^2\boldsymbol{I}$.

As mentioned in §§ 1 and 2, averaged equations like (2.1) have been given by various writers on the subject. One can distinguish between investigations like those by Ishii (1975) or Buyevich & Shchelchkova (1978), where only formal equations are given, and studies in which the averaging is actually carried out. An example of the first category is the book by Ishii (1975). A formal expression is given there for the so-called mixture volumetric momentum source $\boldsymbol{M}_\mathrm{M}$, which can be obtained by addition of the two interaction terms in the momentum equations in (2.1) of the present paper:

$$\boldsymbol{M}_\mathrm{M}=\frac{2\gamma}{R}\nabla\alpha+\boldsymbol{M}_\mathrm{M}^\mathrm{H}.$$

$\boldsymbol{M}_\mathrm{M}^\mathrm{H}$ is a force arising from deviations of the bubble shape from the spherical form, due to bubble motion. Comparing this with our result (4.5), we would obtain for $\boldsymbol{M}_\mathrm{M}$

$$\boldsymbol{M}_\mathrm{M}=\nabla\frac{2\gamma\alpha}{R}-\nabla\cdot\alpha\rho_\ell[\tfrac{3}{20}\{|\boldsymbol{U}_\mathrm{g}-\boldsymbol{U}_0|\}^2\boldsymbol{I}-\tfrac{9}{20}(\boldsymbol{U}_\mathrm{g}-\boldsymbol{U}_0)(\boldsymbol{U}_\mathrm{g}-\boldsymbol{U}_0)].$$

There are only a few studies known to the present authors in which the averaging is actually carried out. They are mostly published by Russian investigators and almost all of them (e.g. Nigmatulin 1979) use so-called cell models in dealing with the interactions. In these models the suspension is divided up into cells. In the centre of a cell is a test particle, and the flow relative to the test particle is calculated under certain assumptions about the prevailing boundary conditions along the walls of the cell. The difficulty with cell models is that their accuracy is not known. It is therefore difficult to assess the validity of the results. Since, apart from this, the derivations mentioned above contain numerous additional assumptions, we cannot give a more or less precise evaluation of how differences between momentum equations in these works and our momentum equation (6.1) are produced. We can only report that the term $(\mathrm{d}R/\mathrm{d}t)^2\boldsymbol{I}$ does not occur in these studies and that the non-deviatoric part associated with $\boldsymbol{U}_\mathrm{g}-\boldsymbol{U}_0$ does occur, albeit in a different form. Voinov & Petrov (1977), for example, find (they do not use a cell model) as momentum equations, neglecting as we did the contribution of gas to the momentum,

$$\frac{\partial}{\partial t}\rho_\ell(1-\alpha)\,\boldsymbol{U}_\ell+\nabla\cdot\rho_\ell(1-\alpha)\,\boldsymbol{U}_\ell\,\boldsymbol{U}_\ell$$
$$=-\nabla\langle p\rangle+\nabla\cdot\tfrac{1}{2}\rho_\ell\,\alpha\{|\boldsymbol{U}_\mathrm{g}-\boldsymbol{U}_\ell|\}^2\boldsymbol{I}-\nabla\cdot\tfrac{1}{2}\rho_\ell\,\alpha(\boldsymbol{U}_\mathrm{g}-\boldsymbol{U}_\ell)(\boldsymbol{U}_\mathrm{g}-\boldsymbol{U}_\ell).$$

7. On the relative motion between the phases

The momentum equation (6.1) together with mass-conservation equations has to be supplemented with additional relations to complete the set of equations. For most purposes the liquid can be considered as incompressible and isothermal. For the gas phase an energy equation has to be formulated which, in the simple case of isothermal behaviour, and excluding coalescence or breakup of bubbles, can be put in the form

$$\rho_g R^3 = \text{constant}, \quad \frac{p_g}{\rho_g} = \text{constant}. \tag{7.1}$$

Inserting this in (4.7) makes this relation in fact an equation that relates the average bubble radius to the bulk pressure and the relative velocity $\boldsymbol{U}_g - \boldsymbol{U}_0$.

The mass-conservation equations like (2.4) and (2.5) are accompanied in the present theory by one momentum equation, (6.1). Since this is not sufficient, we have to look at the equation of motion of one of the two phases separately. We choose for this the gas phase. As we shall see, the difficulties associated with complex characteristics are associated with erroneous equations for the gas phase. At the present stage of approximation, in which the inertia of the gas is negligible, we have to assume that the average force on bubbles is zero,

$$n\langle \boldsymbol{F} \rangle = 0. \tag{7.2}$$

The averaging is, as before, over all realizations of N bubbles in a volume V. The volume V is centred around the location $\boldsymbol{x}$ in the suspension. For the suspension to be statistically homogeneous, it is required that *averaged quantities are constant in* V. Such quantities are $\boldsymbol{U}_0(\boldsymbol{x}, t)$, $\boldsymbol{U}_g(\boldsymbol{x}, t)$ and $\boldsymbol{U}_\ell(\boldsymbol{x}, t)$. The averaging concerns quantities that vary rapidly as a result of the motion of a test bubble (particle) or of interaction between bubbles. Such interactions are neglected in the present approximation, and we can envisage a bubble moving with velocity $\boldsymbol{U}_g(\boldsymbol{x}, t)$ through a liquid which has far from the bubble the velocity $\boldsymbol{U}_0(\boldsymbol{x}, t)$. This is the situation envisaged in figure 3 and described by the potential ϕ in (3.10). Averaging is over the variable $\boldsymbol{r}$, and all quantities that are constant on the scale of $\boldsymbol{r}$ remain constant in the averaging process.

The motion of a closed body through uniform flow of a perfect liquid is a classic subject in fluid mechanics, for which the fluid impulse (Batchelor 1967, p. 408) is a useful concept. The impulse is the product of the virtual mass, m say, and the relative velocity, here $\boldsymbol{U}_g - \boldsymbol{U}_0$. For a sphere with volume τ the virtual mass is $\frac{1}{2}\rho_\ell \tau$ in an infinite medium. For a massless sphere the time rate of change of the impulse equals the external force. In our case this is the pressure force on the bubble due to the time rate of change of the liquid velocity, $\rho_\ell \tau \partial \boldsymbol{U}_0/\partial t$. Thus the average momentum equation for the gas phase is

$$\frac{\partial}{\partial t}\tfrac{1}{2}\rho_\ell \tau\{\boldsymbol{U}_g(\boldsymbol{x}, t) - \boldsymbol{U}_0(\boldsymbol{x}, t)\} = \rho_\ell \tau \frac{\partial \boldsymbol{U}_0}{\partial t}. \tag{7.3}$$

We can make this relation more realistic by the addition of effects of viscosity and gravity. As mentioned in §3, the drag on the bubble can be found from calculation of the dissipation. The result is, in the case of a uniform relative velocity $\boldsymbol{U}$,

$$\boldsymbol{D} = 12\pi\mu R \boldsymbol{U},$$

where μ is the viscosity of the liquid. Including this viscous drag and a buoyancy force, the relation (7.3) becomes

$$\frac{\partial}{\partial t}\tfrac{1}{2}\rho_\ell \tau\{\boldsymbol{U}_g - \boldsymbol{U}_0\} = \rho_\ell \tau \frac{\partial \boldsymbol{U}_0}{\partial t} - 12\pi\mu R(\boldsymbol{U}_g - \boldsymbol{U}_0) + \rho_\ell \tau \boldsymbol{g}. \tag{7.4}$$

Within the present approximation, (7.4) is the correct momentum equation of the gas. If an expression is required that is valid for larger values of the concentration

α, interactions between bubbles have to be taken into account. For instance in an approximation correct to $O(\alpha^2)$, interactions between two bubbles have to be considered. This means that for the added mass in an expression like

$$m = \tfrac{1}{2}\rho\tau(1+b\alpha) \tag{7.5}$$

the coefficient b has to be determined. This has been done, with the use of Batchelor's renormalization technique (see e.g. Batchelor 1974) to avoid divergence difficulties, in van Wijngaarden (1976*b*). We take this opportunity to report that the value $b = 2.78$, as reported there, is incorrect owing to a computational error, and should be $b = 3.32$. A similar calculation of the frictional force as a function of α is the subject of current work by us.

Since much has been written on the subject of the correct form which the equation of motion for the gas should take (see e.g. van Wijngaarden 1976*a*), it seems appropriate to devote some discussion to it. The discussion in the literature has been centred around the question of how the classical expression for the motion of a bubble in a uniformly accelerated fluid could be extended to spatially nonuniform flow. Various forms have been proposed in which material derivatives with respect to the liquid as well as with respect to the gas occur. A discussion on the correct form of the pertinent equation for non-uniform flow is outside the scope of the present paper. Of importance in the context of this investigation is to note that, in fact, such an equation is not needed here, because, in the averaging procedure, average quantities are constant in the averaging volume and only time derivatives are needed, as shown by (7.3).

8. Equations of motion for one-dimensional vertical flow

In this section we summarize the equations derived in the previous sections for one-dimensional transient motion. In front of the equations the pertinent number is indicated, as it appears in the foregoing sections. Gravity is pointing in the negative x-direction.

$$(2.4)\quad \frac{\partial}{\partial t}\rho_g\,\alpha+\frac{\partial}{\partial x}(\rho_g\,\alpha U_g) = 0, \tag{8.1}$$

$$(2.5)\quad \frac{\partial}{\partial t}(1-\alpha)+\frac{\partial}{\partial x}U_\ell(1-\alpha) = 0, \tag{8.2}$$

$$(6.1)\quad \rho_\ell(1-\alpha)\left\{\frac{\partial U_\ell}{\partial t}+U_\ell\frac{\partial U_\ell}{\partial x}\right\} = -\frac{\partial}{\partial x}\langle p\rangle-\frac{\partial}{\partial x}\alpha\rho_\ell\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2+\tfrac{1}{4}(U_g-U_0)^2\right]-\rho_\ell(1-\alpha)\,g, \tag{8.3}$$

$$(7.4)\quad \frac{1}{2}\frac{\partial}{\partial t}\rho_\ell\,\tau(U_g-U_0) = \rho_\ell\,\tau\frac{\partial U_0}{\partial t}-12\pi R\mu(U_g-U_0)+\rho_\ell\,\tau g, \tag{8.4}$$

$$(5.4)\quad (1-\alpha)\,U_\ell+\alpha U_g = U_0, \tag{8.5}$$

$$(7.1)\quad \rho_g R^3 = \text{constant},\quad \frac{p_g}{\rho_g} = \text{constant}, \tag{8.6}$$

$$(4.6)\quad \langle p\rangle = (1-\alpha)\langle p\rangle_\ell + \alpha\langle p\rangle_g - \frac{2\gamma}{R}\alpha, \tag{8.7}$$

$$(4.7)\quad \langle p\rangle_g = p_g = \langle p\rangle + \frac{2\gamma}{R} + \rho_\ell\left[\frac{3}{2}\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2} - \tfrac{1}{4}(U_g - U_0)^2\right]. \tag{8.8}$$

It should be recalled that these equations are correct in the first-order terms in α.

9. Acoustic waves and concentration waves

Before inspecting the characteristics of the system (8.1)–(8.8), it is useful to discuss two types of waves that can propagate through a bubbly flow and which are related to the concept of characteristics. We recall that according to the theory of partial differential equations of hyperbolic type, discontinuities travel with the characteristic speed.

One such discontinuity is a sound wave. Acoustic waves in bubbly flows have been extensively studied. The speed of sound c_f in a *quiescent* mixture is given by

$$c_f^2 = \frac{p}{\rho_\ell\,\alpha(1-3\alpha)}. \tag{9.1}$$

The way in which acoustic waves are affected by relative translational motion, relative radial motion, viscosity and other effects is rather well established, and we may refer here to reviews as for example in Lauterborn (1980). For further reference we note here only that, for the case in which viscosity is so large that in the sound wave the bubbles move with the liquid, the sound velocity is slightly lower than in (9.1) and given by

$$c_0^2 = \frac{p}{\rho_\ell\,\alpha(1-\alpha)}. \tag{9.2}$$

Another type of wave is related to kinematic waves or continuity waves. These are waves in which disturbances in the void fraction α propagate through the bubbly flow, thereby causing disturbances in the velocities of fluid and gas but leaving the pressure essentially unchanged. For traffic waves (Whitham 1974), which form the classical example of kinematic waves, there exists an empirical relation between traffic velocity (comparable to U_g) and traffic density (comparable to α). Here the likeness between traffic waves and our type of waves ceases, because here the relations that we need to express U_g in α, (8.4) for example, contain inertia terms. Only such waves are kinematic waves, properly speaking, in which inertia does not take part. We shall, because of this difference, term our waves concentration waves.

Concentration waves of small and of finite amplitude have been observed and studied under laboratory conditions as reported in Bernier (1981) and in Bouré & Mercadier (1982). Equations that are supposed to describe their behaviour should therefore possess real characteristics. Complex characteristics lead to instability, and stable waves like those observed could not exist. That is why one must stipulate real characteristics. It will turn out in §10 that the system of equations summarized in §8 has real characteristics, whereas in the literature complex characteristics have been found, notably because of a different form of (8.5).†

Properties of concentration waves in our theory are easily obtained under

† One exception is the relation chosen in Prosperetti & van Wijngaarden (1976) in which the time derivatives in (8.4) are taken as material derivatives associated with the bubbles. In retrospect, this is not correct.

simplified conditions. Upon neglecting compressibility, the mass-conservation equation (8.1) for the gas phase is reduced to

$$\frac{\partial \alpha}{\partial t}+U_{\mathrm{g}}\frac{\partial \alpha}{\partial x}+\alpha\frac{U_{\mathrm{g}}}{\partial x}=0. \tag{9.3}$$

If we leave out the viscous term in (8.4) for convenience and also leave out gravity, we have from (8.4)

$$U_{\mathrm{g}}=3U_0(t). \tag{9.4}$$

Note that, at constant ρ_{g}, addition of (8.1) and (8.2) shows together with (8.5) that U_0 depends on time only. Inserting (9.4) in (9.3) gives directly

$$\frac{\partial \alpha}{\partial t}+U_{\mathrm{g}}\frac{\partial \alpha}{\partial x}=0. \tag{9.5}$$

This shows that under the above conditions concentration waves travel at the velocity of the gas.

The same result is obtained when we consider bubbles rising in a vertical pipe, buoyancy being balanced by viscous drag:

$$\tfrac{4}{3}\pi\rho_\ell R^3 g=12\pi\mu R(U_{\mathrm{g}}-U_0). \tag{9.6}$$

Here too U_{g} depends only on U_0, and (9.5) follows again.

The observations by Bernier (1981) and Bouré & Mercadier (1982) show that the speed of propagation of concentration waves depends on α, whereas in the above examples they travel at U_{g}. The reason for this discrepancy is that we have neglected interactions between the bubbles. To obtain quantitative effects of α on the speed of propagation of concentration waves we shall have to incorporate interactions. This can be illustrated with the traffic waves. When drivers (cars may be compared to bubbles) are not aware of the presence of other cars, and the external conditions along the road do not change, all drivers will move with a speed compatible with these external conditions. Only when the drivers let themselves be influenced by the presence of others will they adapt their speed to the traffic density, and then traffic waves with a speed depending on density may occur.

10. Characteristic velocities

After this discussion of acoustic waves and concentration waves we finally turn our attention to the characteristics of the system (8.1)–(8.8). In order to leave out frequency dispersion we take $\langle p\rangle=\langle p\rangle_{\mathrm{g}}=p$. For small dispersion and small viscosity the equations of motion can be reconciled with the concept of characteristics as in single-phase gases, as described in Lighthill (1956) or Whitham (1974). We obtain from (8.1)–(8.8)

$$\frac{\partial}{\partial t}(\alpha\rho_{\mathrm{g}})+\frac{\partial}{\partial x}(\alpha\rho_{\mathrm{g}}U_{\mathrm{g}})=0,$$

$$\frac{\partial}{\partial t}(1-\alpha)+\frac{\partial}{\partial x}(1-\alpha)\,U_\ell=0,$$

$$\rho_\ell(1-\alpha)\left\{\frac{\partial}{\partial t}U_\ell+U_\ell\frac{\partial U_\ell}{\partial x}\right\}+\frac{\partial p}{\partial x}=-\rho_\ell(1-\alpha)\,g,$$

$$\frac{\partial}{\partial t}\{U_{\mathrm{g}}\tau+\tfrac{3}{2}(U_0-U_{\mathrm{g}})\tau\}-U_0\frac{\partial \tau}{\partial t}=12\pi\nu R(U_{\mathrm{g}}-U_0)-\tau g.$$

The characteristic roots are found by inserting

$$\frac{\partial}{\partial t} = -\lambda \frac{\partial}{\partial x}$$

in these equations and solving for λ. One root is $\lambda = 0$ and the three other ones are roots of the algebraic equation

$$(\lambda - U_\ell)\{\tfrac{1}{2}\rho_\ell \alpha(1-3\alpha)(U_g-\lambda)(U_\ell-\lambda)+\tfrac{3}{2}p\alpha-\tfrac{1}{2}\rho_\ell\alpha(1-\alpha)(U_g-U_\ell)(U_\ell-\lambda)\} \\ -p\{\tfrac{3}{2}\alpha(U_\ell-U_g)-\tfrac{1}{2}(1-3\alpha)(U_g-\lambda)\} = 0. \tag{10.1}$$

We write
$$\xi = \frac{\lambda - U_\ell}{c_f}, \quad \eta = \frac{U_g - U_\ell}{c_f}, \tag{10.2}$$

c_f being the sound velocity defined in (9.1). With these, (10.1) becomes

$$\xi^3 - \eta\left(1+\frac{1-\alpha}{1-3\alpha}\right)\xi^2 - \xi + \eta = 0. \tag{10.3}$$

In practice η is a small quantity because c_f, although very much smaller than the sound velocity in pure liquid, is of order 10^2 m/s and the velocities of either gas or liquid are in most cases of the order of 1 m/s. If we write

$$\xi = \xi_0 + \eta\xi_1 + \eta^2\xi_2 + \dots$$

and expand the left-hand side of (10.3) in ascending powers of η, we obtain when we neglect terms of order η^2

$$(\xi_0^3 - \xi_0) + \eta\{3\xi_0^2\xi_1 - \left(1+\frac{1-\alpha}{1-3\alpha}\right)\xi_0^2 - (\xi_1 - 1)\} = 0.$$

Solving for ξ_0 and ξ_1, we find

$$\xi = 0, \quad \text{with} \quad \xi = 1, \tag{10.4}$$

$$\xi_0^2 = 1, \quad \text{with} \quad \xi_1 = \frac{1-\alpha}{2(1-3\alpha)}. \tag{10.5}$$

Using (10.2) and returning to physical variables, we finally obtain for the roots of (10.1)

$$\text{from (10.4):} \quad \lambda_1 = U_g + O\left\{\left(\frac{U_g - U_\ell}{c_f}\right)^2\right\}, \tag{10.6}$$

$$\text{from (10.5):} \quad \lambda_{2,3} = \tfrac{1}{2}(U_g + U_\ell) - \frac{1}{2}\left(1-\left(\frac{c_f}{c_0}\right)^2\right)(U_g - U_\ell) \pm c_f + O\left\{\left(\frac{U_g - U_\ell}{c_f}\right)^2\right\}, \tag{10.7}$$

in which c_f and c_0 are given by (9.1) and (9.2) respectively.

First of all, we see that all λs are real. Further, from general acoustic theory, we infer that λ_2 and λ_3 are associated with sound waves. In a single-phase fluid the characteristic speeds are $U \pm c$, U being the fluid velocity and c the sound velocity. In a quiescent bubbly suspension sound is propagated with speed c_f. In a moving fluid where $U_\ell = U_g = U_0$ sound is convected with the flow, see (10.7), as in a single-phase fluid. However, when both phases move at different velocities, the characteristic speeds depend both on the velocities and on α (through the factor $(c_f/c_0)^2$ in (10.7)).

It is interesting to note that, because, from (9.1) and (9.2),

$$\left(\frac{c_f}{c_0}\right)^2 \to 1 \quad \text{as} \quad \alpha \to 0,$$

we have

$$\lambda_{2,3} \to \tfrac{1}{2}(U_g + U_\ell) \pm c_f \quad \text{as} \quad \alpha \to 0. \tag{10.8}$$

This behaviour for $\alpha \to 0$ of the characteristic velocities was also found by Prosperetti & van Wijngaarden (1976), who, however, used an erroneous equation for the relative velocity (see footnote on p. 314). Finally, the characteristic speed λ_1 in (10.6) is associated with concentration waves. We have already emphasized that, because no interactions between the bubbles are taken into account, the kinematic wave velocity does not show a dependence on the void fraction. This dependence will only become apparent after inclusion of interaction effects.

In the above analysis the characteristics have been evaluated for small η. It can be shown that the roots of (10.3) are also real for arbitrary values of η.

11. Conclusion

A system of equations (8.1)–(8.8) for bubbly flows has been obtained in which fluctuations are taken into account. The fluctuations are of purely hydrodynamic nature. The system (8.1)–(8.8) is consistent in two ways. First, all equations have been obtained by ensemble averaging over all possible realizations of a large number of bubbles in a given volume. Secondly, in the derivations it is assumed that the suspension is dilute and that the velocity fluctuation at a given point is due to the motion of one bubble only. (The probability of a second bubble being at a distance of the order of the bubble radius is of the order of the concentration squared.)

It appears that the system (8.1)–(8.8) is completely hyperbolic. This is a remarkable result because equations that have been previously derived by others, along different lines, appear to have complex characteristics, making the solutions depend in a discontinuous way on the initial conditions. It is shown that two characteristics are associated with sound waves. One is zero (at least in our approximation, in which in the momentum equations the density of the gas in the bubbles is neglected), and the remaining one is shown to be associated with concentration waves. In the present approximation of no interactions these travel with the gas velocity, which is precisely the characteristic root found in §10.

This research is supported by the Foundation F.O.M. under contract TTN 22.0213. The authors thank B. H. A. A. van den Brule, who made, during the preparation of his Master's thesis, valuable contributions to the material in §§3–5.

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